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Derivative Asset Analysis in Models with Level-Dependent and Stochastic Volatility

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In this survey we discuss models with level-dependent and stochastic volatility from the viewpoint of derivative asset analysis. Both classes of models are generalisations of the classical Black-Scholes model; they have been developed in an effort to build models that are flexible enough to cope with the known deficits of the classical Black-Scholes model. We start by briefly recalling the standard theory for pricing and hedging derivatives in complete frictionless markets and the classical Black-Scholes model. After a review of the known empirical contradictions to the classical Black-Scholes model we consider models with level-dependent volatility. Most of this survey is devoted to derivative asset analysis in stochastic volatility models. We discuss several recent developments in the theory of derivative pricing under incompleteness in the context of stochastic volatility models and review analytical and numerical approaches to the actual computation of option values.

1. Introduction

Over the last 20 years the classical Black-Scholes model has proven to be a very effective tool for the valuation and the risk-management of derivative securities, and even today most of the trading activity on markets for equity and currency options is based on this model. Nonetheless, in recent years a number of empirical observations have been compiled that are difficult to reconcile both with the assumptions the model imposes on the price process of the underlying asset and with the predictions the model makes on the behaviour of option prices. To mention only a few of these issues that currently mark many debates in derivative asset analysis, most time series of asset returns are said to exhibit "excess kurtosis" and "fat tails" and on options markets we encounter "smile" or "skew" patterns of implied volatility.

Researchers have therefore attempted to build new option pricing models that are flexible enough to cope with these empirical facts. A good deal of this research concentrates on relaxing the stringent and unrealistic assumption of constant volatility imposed on the price process of the underlying security. Basically the continuous-time approaches to a more refined volatility modelling can be divided into two classes. The *deterministic volatility* models (DV-models) take the volatility to be a function of the price level of the underlying security whereas the *stochastic volatility* models (SV-models) assume that the volatility is itself given by a stochastic process that is only imperfectly correlated with the asset price process.

In this survey we discuss both approaches from the viewpoint of derivative asset analysis. We are mainly concerned with the derivation and computation of prices and hedge portfolios for options and with an analysis of their qualitative properties. Our focus is on models which are set up in continuous time, but we briefly treat discrete-time GARCH-models as these are both helpful tools for the estimation of continuous-time models and interesting models in their own right.

We start our analysis in Section 2 by briefly recalling the standard theory for pricing and hedging derivatives in complete and frictionless markets and apply this theory in the framework of the Black-Scholes model. This Section serves several purposes. It is a quick introduction for the "newcomer" to the field. Moreover, it illustrates that option pricing formulae such as the celebrated Black-Scholes formula hinge on several strong assumptions on the price processes of the underlying assets. To the extent that these are violated simply applying the recipes of standard option pricing theory may lead to answers which are nonsensical from an economic viewpoint and which may have severe consequences for the practioner who does not take the necessary care when applying a particular option pricing model.

In Section 3 we discuss more thoroughly the empirical evidence contradicting underlying assumptions and predictions of the classical Black-Scholes model. We then go on and study the class of deterministic volatility models in Section 4. Here we are mainly interested in the so-called implied deterministic volatility models proposed for instance by Dupire [23] and Rubinstein [63]. In these models one tries to determine a volatility function for the price process of the underlying asset in order to "fit" the prices of traded option contracts. The models thus obtained can then be used for the pricing and hedging of more complex derivatives. The main virtue of this class of models is *completeness*;[1] conceptual difficulties concerning the way options should be priced and hedged do therefore not arise in this framework.

As most of the recent extensions of the classical Black-Scholes model belong to the class of stochastic volatility models, we devote the greatest part of this survey to the study of derivative asset analysis in this class of models. We start by introducing different popular specifications from the recent literature

[1] A model for the price process of the underlying security is termed complete if every derivative contract can be replicated by a dynamic trading strategy.

in Section 5. We introduce certain GARCH-models and discuss the convergence of GARCH-models to continuous-time diffusions.

In Section 6.1 we consider the conceptual problems for derivative asset analysis in SV-models. We show that there is a price to pay for the increase in realism gained by modelling volatility as a stochastic process: stochastic volatility models are typically incomplete, and there are many possible price processes for an option that are consistent with no arbitrage. We demonstrate by means of an example that arbitrage pricing alone may be of no help when it comes to restricting the range of possible option prices. This casts some doubts on the stochastic volatility option pricing models proposed in the Finance literature where one of the price processes consistent with no arbitrage is picked more or less ad hoc. Moreover, the problem of hedging derivatives is not adressed in this literature. As this is a key issue for practioners we review two approaches to derivative asset analysis in incomplete markets that are based mainly on hedging arguments, namely *superreplication* as introduced by El Karoui and Quenez [27] and *(local) risk minimization* as developed in Fölmer and Schweizer [33] and related papers.

In the theory of superreplication one seeks to find the cheapest selffinancing trading strategy that yields a terminal payoff no smaller than the payoff of the derivative one wants to cover. Using a deep result from El Karoui and Quenez [27] we show that in the case of certain SV-models with unbounded volatility there exists only a trivial superreplication strategy. Therefore, at least for these models, superreplication in the sense of El Karoui and Quenez [27] does not seem to be a viable approach to the hedging of derivatives.

The theory of local risk minimization seeks to determine a trading strategy in the underlying asset that reduces the risk of a derivative to its "intrinsic component". While the "hedgeable part" of a derivative can be priced by standard replication arguments, economic equilibrium arguments or concepts from insurance pricing are needed to find a price of the "cost process" that represents the intrinsic risk of the derivative. We show that the general recipe for the computation of locally risk minimizing hedging strategies given in Fölmer and Schweizer [33] is easily applied to SV-models and yields very intuitive results. In doing so we moreover take the chance to correct a minor error of Hofmann, Platen, and Schweizer [46].

Section 6.1 forms the core of this paper. It presents a view on derivative analysis in SV models that is rarely taken in the literature on option pricing under stochastic volatility and it contains in addition some new results.

We discuss several approaches to the actual computation of option prices in section 6.2. Here we review both, theoretical and analytical approaches. We conclude our analysis of option pricing under stochastic volatility by collecting evidence on the qualitative behaviour of option prices in SV-models. It turns out that these prices exhibit the same qualitative properties than do the observed prices of traded option contracts. This gives some hope that SV models might be a useful tool for the risk-management of derivatives.

We do not devote much attention to the estimation of SV-models and sketch only one possible approach. This is not meant to imply that this is an uninteresting or unimportant topic, it is simply due to the fact that the author's field of expertise lies elsewhere. For further information on the estimation of SV-models and more generally the estimation of diffusion models from discrete observations see for example DACUNHA-CASTELLE and FLORENS-ZMIROU [14] or the survey articles GÖING [39], SHEPARD [66], AIT-SAHALIA [1] and GHYSELS, HARVEY, and RENAULT [38]. Another interesting survey on option pricing under stochastic volatility is HOBSON [44]. As in our paper the latter article considers mainly the pricing of options in SV-models. The author does not address the issue of option pricing under incompleteness in great detail, but he is more explicit about the estimation of SV-models than we are; in particular this article contains an excellent bibliography on this subject.

2. Pricing and Hedging Derivatives — Standard Theory

We start our survey by briefly reviewing the standard theory for the pricing of derivative securities such as options. Our exposition largely follows FÖLLMER [32]. We consider a market where a risky asset, in the sequel simply referred to as the stock, and some riskless bond or money market account B are traded. The price fluctuations of the stock are described by some stochastic process $X = (X_t)_{0 \le t \le \infty}$ which is defined on some filtered probability space $(\Omega, \mathcal{F}, P)$, $(\mathcal{F}_t)_{t \ge 0}$. For simplicity we assume that $B_t = 1$ for all $t \ge 0$.[2] A typical model for the price process of the risky asset is the *generalized Black-Scholes model* where X is given by the solution of the following SDE

$$dX_t = \mu(t, X_t) X_t dt + \sigma(t, X_t) X_t dW_t \,. \tag{1}$$

Here W is a standard Brownian Motion on $(\Omega, \mathcal{F}, P)$ and μ and σ are sufficiently smooth such that there is a unique solution to (1) which is moreover strictly positive. When talking of a generalized Black-Scholes model we shall always assume that $(\mathcal{F}_t)_{t \ge 0}$ is the filtration generated by the Brownian motion W. The model (1) has the following intuitive interpretation: at a given point in time $\mu(t, X_t)$ describes the instantaneous growth rate of the asset, while the *volatility* $\sigma(t, X_t)$ measures the instantaneous variance of the return process $\ln X$. Hence $\sigma(t, X_t)$ can be interpreted as (local) measure of the risk incurred by investing one unit of the money market account into the stock. In case that $\sigma(t, x)$ is a constant or at most a function of time model (1) is termed the *classical Black-Scholes model.*

Now imagine an investor such as a bank who considers selling a contingent claim, i.e. a $\mathcal{F}_T$-measurable random variable $\bar{H}$. $\bar{H}$ is interpreted as payoff at date T of some financial contract. Typically $\bar{H}$ is a derivative asset, i.e. the value of $\bar{H}$ is determined by the realisation of the price path of X. The most

[2] This assumption does not exclude nonzero interest rates from our analysis, if we interprete X as *forward price process* of the stock, i.e. if we choose the bond as numeraire. For a general discussion of the role of numeraires in derivative asset pricing theory see for instance EL-KAROUI, GEMAN and ROCHET [25].

popular examples are European call and put options with maturity date T and exercise price K where $\bar{H} = [X_T - K]^+$ or $\bar{H} = [K - X_T]^+$, respectively. As $\bar{H}$ is typically unknown at date $t = 0$ such a contingent claim constitutes a risk. Therefore two questions arise for our investor: How should he price the claim and how should he deal with the risk incurred by selling the contract? The "modern" answer to these questions dates back to the seminal papers by Black and Scholes [7] and Merton [55]. They showed that under certain assumptions the payoff of a derivative security can be replicated by a dynamic trading strategy in the underlying asset, such that its risk can be eliminated. This concept of dynamic hedging, which can be carried over to more sophisticated models than (1), and not the celebrated Black-Scholes formula which holds only in the classical Black-Scholes model should be viewed as major contribution of these papers.

Let us now explain their argument in more detail. Assume that the asset price process X admits an *equivalent martingale measure* Q, i.e. a probability measure with the same nullsets as P such that X is a Q-martingale. This assumption excludes arbitrage opportunities from our model.[3] Moreover it ensures that X is a P-semimartingale such that we may define stochastic integrals with respect to X.

Now consider a dynamic trading strategy (ξ, η) where ξ_t gives the amount held in the risky asset at time t and η_t gives the position in the bond. Of course our position at t should depend only on information available up to time t, that is we require ξ to be *predictable* and η to be *adapted* with respect to $(\mathcal{F}_t)_{t\geq 0}$. At time t the value of our hedge portfolio equals

$$V_t = \xi_t X_t + \eta_t \,. \tag{2}$$

As $B_t \equiv 1$ the cumulated *gains from trade* of following this strategy up to time t are measured by the stochastic integral $\int_0^t \xi_s dX_s$. Hence the *cumulative cost* C_t from following this strategy up to time t is given by

$$C_t = V_t - V_0 - \int_0^t \xi_s dX_s \,. \tag{3}$$

The strategy will be called *selffinancing* if the cumulative cost is zero, i.e. if we have

$$V_t = V_0 + \int_0^t \xi_s dX_s \quad \text{for all } 0 \leq t \leq T \,. \tag{4}$$

Suppose now that our contingent claim can be represented as a stochastic integral with respect to X, i.e. $\bar{H} = H_0 + \int_0^T \xi_s^H dX_s$. Then we may construct a dynamic hedging strategy for H as follows. Define

[3] Roughly speaking an arbitrage opportunity is a selffinancing strategy with zero initial investment and a nonnegative value process V_t with $P[V_T > 0] > 0$. Absence of arbitrage opportunities is known to be "essentially equivalent" to the existence of an equivalent martingale measure. There is a long literature on this subject starting with Harrison and Kreps [40] and culminating in Delbaen and Schachermayer [15].

$$\xi = \xi^{\bar{H}} \text{ and } \eta_t = H_0 + \int_0^t \xi_s^{\bar{H}} dX_s - \xi_t^{\bar{H}} X_t \,. \tag{5}$$

This strategy is selffinancing with value process $V_t^{\bar{H}} = H_0 + \int_0^t \xi_s^{\bar{H}} dX_s$. In particular $V_T^{\bar{H}} = \bar{H}$. Therefore, at any time $t \le T$ we can replicate the claim by starting with an investment of $V_t^{\bar{H}}$ and following the above strategy. There are no further payments and hence no further risks. This implies that at time t the *fair price* of the claim should be equal to $V_t^{\bar{H}}$.

HARRISON and PLISKA [41] proposed the following shortcut to computing $V_t^{\bar{H}}$. Under certain integrability conditions the stochastic integral $\int_0^t \xi_s^{\bar{H}} dX_s$ is a Q-martingale and hence

$$E^Q \left[\int_t^T \xi_s^{\bar{H}} dX_s \mid \mathcal{F}_t \right] = 0 \quad \text{for all } t \,.$$

This yields the so-called *risk-neutral pricing rule* for the claim $\bar{H}$

$$H_t := V_t^{\bar{H}} = E^Q[\bar{H} \mid \mathcal{F}_t] \,; \tag{6}$$

in particular the fair price process $H = (H_t)_{0 \le t \le T}$ is a Q-martingale. Harrison and Pliska moreover showed that the market is *complete*, i.e. every Q-integrable claim admits a representation as stochastic integral with respect to X, if and only if there is only one equivalent martingale measure for X.

This elegant approach to pricing and hedging contingent claims hinges on several crucial hypotheses. Obviously if our argument is to work for all claims the market must be complete. Moreover, in our definition of the gains from trade we implicitely assumed that there are no market frictions such as taxes and transaction costs, and that our potential seller is "small" compared to the size of the market, meaning that the implementation of his hedging strategy doesn't affect the price process of the stock. Much of the recent research in Finance has concentrated on relaxing these assumptions. We will survey recent approaches to pricing and hedging of derivatives in incomplete markets in the course of our analysis of stochastic volatility models in section 6.1. A representative example of recent work on transactions costs is BENSAID, LESNE, PAGES, and SCHEINKMAN [4]; the pricing and hedging of options in markets with a large trader is for instance studied by JARROW[49] or FREY and STREMME [37] and FREY [35].

Let us now apply the above approach to pricing and hedging derivatives in the context of the generalized Black-Scholes model (1). Define

$$G_T := \exp\left(-\int_0^T [\mu(t, X_t)/\sigma(t, X_t)] dW_t - \frac{1}{2} \int_0^T [\mu(t, X_t)/\sigma(t, X_t)]^2 dt \right),$$

Under some integrability conditions we have $E[G_T] = 1$. In that case we may define a new probability measure Q on $\mathcal{F}_T$ by putting $dQ/dP := G_T$. According to Girsanov's theorem[4] the process X solves under Q the SDE

[4] For an account of Girsanov's theorem for Brownian motion and sufficient conditions for $E[G_T] = 1$ see for instance KARATZAS and SHREVE [51, Section 3.5].

$$dX_t = \sigma(t, X_t) X_t dW_t^Q \tag{7}$$

for the Q-Brownian motion $W_t^Q := W_t + \int_0^t [\mu(s, X_s)/\sigma(s, X_s)]ds$. Hence X is a local Q-martingale and a martingale under some integrability assumptions.

If the volatility function $\sigma(t, x)$ is strictly positive, market completeness follows from the martingale representation theorem for Brownian motion, see e.g. Karatzas and Shreve [51, Section 3.4 D]. This theorem ensures that for any Q-integrable $\mathcal{F}_T$ measurable random variable $\bar{H}$ the martingale $H_t = E^Q[\bar{H} \mid \mathcal{F}_t]$, $0 \le t \le T$ can be represented as stochastic integral, i.e. there is a predictable process $\psi^{\bar{H}}$ such that $H_t = H_0 + \int_0^t \psi_s^{\bar{H}} dW_s$. If we now define $\xi_s^{\bar{H}} := \psi_s^{\bar{H}} / (\sigma(s, X_s)\, X_s)$ we immediately get

$$\bar{H} = H_0 + \int_0^T \frac{\psi_s^{\bar{H}}}{\sigma(s, X_s) X_s}\, \sigma(s, X_s) X_s dW_s^Q = H_0 + \int_0^T \xi_s^{\bar{H}} dX_s \,.$$

Now there remains of course the task of computing price and hedging strategy. For the purposes of this paper it is enough to consider claims whose payoff has the form $\bar{H} = g(X_T)$.[5] For those derivatives price and hedge portfolio can be computed by means of a parabolic partial differential equation. Denote by $h(t, x)$ the solution of the terminal value problem

$$\frac{\partial}{\partial t} h(t, x) + \frac{1}{2}\sigma^2(t, x) x^2 \frac{\partial^2}{\partial x^2} h(t, x) = 0, \quad h(T, x) = g(x)\,. \tag{8}$$

By Itô's formula we obtain from (8)

$$g(X_T) = h(T, X_T) = h(t, X_t) + \int_t^T \frac{\partial}{\partial x} h(s, X_s) dX_s \,.$$

Hence $\xi_t^{\bar{H}} = \frac{\partial}{\partial x} h(t, X_t)$ and the fair price of the derivative is given by $H_t := h(t, X_t)$.

If we work in the classical Black-Scholes model with only time-dependent volatility and if g equals the payoff of a European option the PDE (8) can be solved explicitely. The usual approach is to transform the problem to the heat equation, see e.g. Willmott, Dewynne and Howison [72, section 5.4]. This yields the famous Black-Scholes formula for European call options:

$$\begin{aligned} h(t,x) &= C_{BS}\,(t\,, x\,, \bar{\sigma}_t)\,, \quad \text{where} \\ C_{BS}(t, x, \bar{\sigma}_t) &= \mathcal{N}(d_t^1) - K\mathcal{N}(d_t^2)\,, \quad \bar{\sigma}_t = (T-t)^{-1} \cdot \int_t^T \sigma^2(s) ds\,, \quad (9) \\ d_t^1 &= \frac{\ln(x/K) + \frac{1}{2}(T-t)\bar{\sigma}_t}{\sqrt{(T-t)\bar{\sigma}_t}}\,, \quad d_t^2 = d_t^1 - \sqrt{(T-t)\bar{\sigma}_t}\,, \quad (10) \end{aligned}$$

and where $\mathcal{N}$ denotes the distribution function of the one-dimensional standard normal distribution. Alternatively one could derive the Black-Scholes formula

[5] For the pricing of path-dependent options in the framework of the classical Black-Scholes model see for instance Willmot, Dewynne, and Howison [72] and the references given therein.

using probabilistic methods to compute the conditional expectation in (6). For an application of this approach in a more general setting see for instance Frey and Sommer [36]. If the volatility is a function of the current price of the risky asset, usually explicit formulas for option prices are no longer available. See however Miltersen Sandmann, and Sondermann [56] for a notable exception and certain applications to the pricing of interest rate derivatives.

3. Empirical Evidence Contradicting the Classical Black-Scholes Model

Over the last 20 years the classical Black-Scholes model has proven to be a very effective tool for the valuation and the risk-management of derivative assets. Nonetheless in the last years a number of observations have been reported which are at odds with both underlying assumptions and predictions of the simple model (1) with constant volatility.

Empirical evidence on the price process of the underlying security suggests that the classical Black-Scholes model does not describe the statistical properties of most financial time series very well. According to this model the return over a short period of time should be normally distributed. Now since the early work of Mandelbrot [54] and Fama [31] researchers have compiled a huge amount of evidence for excess curtosis in financial time series.[6]

A casual observation of financial time series also reveals the presence of volatility clusters, i.e. there are usually periods with high volatility and other periods where volatility is low. This has lead researchers to develop ARCH models which are designed to mimic this behaviour. A brief introduction to these class of models is given in Section 5.2, for a detailed survey see Bollerslev, Chou, and Kroner [9]. As shown in this survey this class of time series models has been applied with great success to financial data. We will see in Section 5.2 that ARCH models can be considered as discrete-time versions of stochastic volatility models. In the latter class of models the volatility itself follows a stochastic process whose innovations are only imperfectly correlated to the stock returns. Hence the success of ARCH-models can be seen as evidence against the assumptions underlying the classical Black-Scholes model. Finally many researchers have found evidence for negative correlation between volatility and stock price movements on equity markets. Following Black [6] this phenomenon is termed the *leverage effect*. Again this evidence contradicts the assumption of constant volatility.

There are also empirical observations on option prices contradicting the predictions of the theory. These observations relate to the behaviour of *implied volatilities*. Suppose we observe that at time t and a given price level $\bar{x}$ of the underlying asset X an option contract is traded at a certain price c. Then we may invert the formula (9) to obtain the implied volatility σ^* of the option. Formally σ^* is the positive solution to the equation $C_{BS}(t, \bar{x}, (\sigma^*)^2) = c$. If the

[6] Roughly speaking this means that the tails of the distribution of the return process are fatter than those of a normal distribution.

stock price process actually followed the classical Black-Scholes model, in an arbitrage-free market these implied volatilities should be independent of exercice price and time to maturity of the traded option contracts and moreover constant over time. However, implied volatilities seem to vary systematically with the exercice price. The implied volatility of "at the money options" (options with $K \approx \bar{x}$) are typically lower than implied volatilities of "out of the money" options or "in the money options". This phenomenon, which was first discovered by RUBINSTEIN [62], is usually termed the "smile pattern" of implied volatility. In many equity markets researchers have also found "skews" that is the implied volatility of put options with $K << \bar{x}$ is significantly higher than the implied volatility of put options with $K >> \bar{x}$. For more evidence on the relation between exercice price and implied volatility of traded option contracts see RUBENSTEIN [62] or TAYLOR and XU [70] and the references given in these papers. Finally implied volatilities also tend to vary stochastically over time; for instance HARVEY and WHALEY [42] have shown that implied volatilities can be described very well by autoregressive models. Again this is at odds with the predictions of the classical Black-Scholes model.

4. Implied Deterministic Volatility Functions: Theory and Empirical Tests

The recent research by DUPIRE [23], DERMAN and KANI [16] and RUBINSTEIN [63] concentrates on building models for the price process of X that can fit a certain observed smile pattern of implied volatility. These models can then be used for the pricing and the hedging of exotic options. Here we will describe the work of Dupire who uses the generalized Black-Scholes model (1) as framework of his analysis.[7] Dupire assumes that at a given point in time t he can observe prices for European call options for all maturity dates $T \geq t$ and all exercice prices $K > 0$. Denote the surface of option prices by $C(K,T)$, $K > 0, T \geq t$ and assume that C is a smooth function. Dupire's aim is to show that there is a unique volatility function $\sigma(t,x)$ such that the observed option prices are consistent with model (1). He argues in two steps.

First he invokes earlier work by BREEDEN and LITZENBERGER [11] to show that the surface $C(K,T)$ determines for all $T > t$ the Lebesgue-density f_T of the distribution of X_T under the risk-neutral measure Q. In fact in our case the risk-neutral pricing rule yields

$$C(K,T) = \int_0^\infty [x-K]^+ f_T(x)dx\,.$$

Differentiating this with respect to K we get $\frac{\partial}{\partial K}C(K,T) = -\int_K^\infty f_T(x)dx$ and hence

$$\frac{\partial^2}{\partial K^2}C(K,T) = f_T(K)\,. \tag{11}$$

[7] Derman and Kani and Rubinstein are developing discrete-time models that are extensions of the binomial model of COX, ROSS, and RUBINSTEIN [13].

As a second step Dupire shows that the function $\sigma(t,x)$ can be inferred from the family $(f_T)_{T>t}$ of density functions using the Kolmogorov forward equation. This is remarkable, as it is well known that in general one-dimensional marginal distributions are not enough to specify the law of a diffusion process. Under some regularity conditions the density function of a diffusion process satisfies the following PDE, which is usually referred to as the *Kolmogorov forward equation*, see for instance KARATZAS and SHREVE [51], equation (5.7.24):

$$\frac{\partial}{\partial T} f_T(K) = \frac{1}{2}\frac{\partial^2}{\partial K^2}\left(a(T,K) f_T(K)\right) - \frac{\partial}{\partial K}\left(b(T,K) f_T(K)\right) , \tag{12}$$

where b is the drift and a is the square of the dispersion coefficient of the diffusion. In our case $b = 0$, as X is assumed to be a Q-martingale. Moreover, $a(T,K) = \sigma^2(T,K) \cdot K^2$. Hence (12) becomes

$$\frac{\partial}{\partial T} f_T(K) = \frac{1}{2}\frac{\partial^2}{\partial K^2}\left(\sigma^2(T,K) K^2 f_T(K)\right) .$$

If we integrate this twice with respect to K and use (11) we obtain

$$\frac{\partial}{\partial T} C(K,T) = \frac{1}{2}\sigma^2(T,K) K^2 \frac{\partial^2}{\partial K^2} C(K,T) + a_1 K + a_2 . \tag{13}$$

Dupire now shows that if the surface of option prices actually stems from an arbitrage-free diffusion model for the underlying price process we must have $\frac{\partial^2}{\partial K^2} C(K,T) > 0$ and $a_1 = a_2 = 0$. Hence we may solve (13) for the volatility function to obtain

$$\sigma^2(T,K) = \frac{2 \frac{\partial}{\partial T} C(K,T)}{K^2 \frac{\partial^2}{\partial K^2} C(K,T)} . \tag{14}$$

To price exotic or American options in this model one may now use numeric methods for PDEs or Monte Carlo simulation as described for instance in DUFFIE [20], chapter 10. As an alternative Dupire and Derman-Kani propose an algorithm to build a discrete-time trinomial tree model incorporating the information contained in the observed option prices.

The great advantage of the above models with implied deterministic volatility (IDV-models) is completeness. Hence they allow for the derivation of hedging strategies and for unique pricing of derivative securities other than call options in a way that is consistent with an observed smile pattern. Unfortunately the IDV-models require an exact observation of call prices for more strikes and maturities than are available on most real options markets. However, this drawback can be overcome by using a parametric form for the volatility function or by using an interpolation algorithm.

The most serious empirical test of the IDV-approach has been carried out by DUMAS, FLEMMING, and WHALEY [22]. They consider the major empirical issue regarding the credibility of the IDV-models, namely the *stability over time* of the "estimated" implied volatility functions: the volatility function derived from an observed surface of option prices at time $t_0 + h$ and the volatility function derived from the option prices observed at time t_0 should (roughly) coincide on their common domain of definition. As they write, "in this case the IDV framework should provide a better means of setting hedge ratios and valuing exotic options. On the other hand, if the function is not stable it cannot be claimed that the true volatility function of the underlying asset has been identified."

DUMAS, FLEMMING, and WHALEY [22] fit different parametric forms for the implied volatility functions to observed prices of exchange-traded S&P 500 index options. Usually they obtain a very good fit.[8] They then use this implied volatility functions to compute the theoretical option prices that should prevail at the spot level of the S&P 500 index one week later if the implied volatility function hadn't changed over time. These values are then compared to the actually observed prices. It turns out that the discrepancy between the observed prices and the prices predicted by the model is relatively large, meaning that at least in this particular case the implied volatility functions are unstable over time. Interestingly they find that this difference between observed and predicted options prices is larger for complex parametrizations of the implied volatility functions than for a constant volatility specification. This is interpreted as evidence that "more complex volatility specifications overfit the observed structure of option prices."

These findings cast some doubts on whether the IDV approach really is an improvement over the traditional (theoretically inconsistent) method of using the Black-Scholes formula with changing volatility. Further testing of this issue is called for. Of course if such research confirms the results of DUMAS, FLEMMING, and WHALEY [22], the task of finding a complete model that is a better risk-management tool than the classical Black-Scholes formula remains an important topic for further investigation. There are several interesting new approaches in this area. BIBBY and SORENSEN [5] propose a model with level-dependent volatility where the asset returns follow approximately a hyperbolic distribution. KALLSEN and TAQQU [50] and HOBSON and ROGERS [45] develop models where the asset price volatility depends on past asset returns. The volatility dynamics in these models are very similar to the volatility dynamics in the celebrated discrete-time GARCH-models, but in contrast to the latter class of models the models of Kallsen-Taqqu and Hobson-Rogers have the virtue of being complete.

5. STOCHASTIC VOLATILITY MODELS

Most of the extensions of the classical Black-Scholes model that have been proposed in recent years belong to the class of *stochastic volatility* (SV) models. Contrary to the approach taken in Section 4, in this class of models the stock price volatility is described by an additional stochastic process whose

[8] Their fit cannot be perfect as the parametric forms they use have less degrees of freedom than there are observed options prices.

innovations are only imperfectly correlated to the stock price process.

5.1. *Continuous-Time Models*

Following HOFMANN, PLATEN, and SCHWEIZER [46] we consider the following Markovian model which is general enough to encompass all the continuous-time stochastic volatility models proposed in the recent literature.

ASSUMPTION 5.1. *The evolution of the stock price X can be described by the following two-dimensional SDE.*

$$dX_t = a(t, X_t, v_t)X_t dt + \sigma(t, X_t, v_t)X_t dW_{1,t} \tag{15}$$

$$dv_t = b(t, X_t, v_t)dt + \eta_1(t, X_t, v_t)dW_{1,t} + \eta_2(t, X_t, v_t)dW_{2,t}\,, \tag{16}$$

where W_1 and W_2 are two independent standard Brownian motions on some probability space $(\Omega, \mathcal{F}, P)$. The filtration $(\mathcal{F}_t)_{t\geq 0}$ is the augmented filtration generated by the Brownian motions. We assume that the SDE (15), (16) has a unique weak solution, whose first component X is moreover strictly positive. Finally $a(t, x, v)$ can be decomposed as

$$a(t, x, v) = \mu(t, x, v)\sigma(t, x, v) \tag{17}$$

for a bounded function μ.

The process v plays the role of an unobservable state-variable that influences the drift and in particular the volatility of the stock price process. We will always assume that η_2 is different from zero, meaning that the state variable is influenced by the second Brownian motion W_2 which is *orthogonal*[9] to the martingale part of X. The instantaneous covariation between X and v is given by

$$\eta_1(t, X_t, v_t)\sigma(t, X_T, v_t)X_t\,;$$

it vanishes if $\eta_1 \equiv 0$.

We now list the stochastic volatility models from the literature and explain how they fit into the above framework. The first stochastic volatility model was proposed by HULL and WHITE [47]. These authors assume that σ_t^2, the square of the volatility, follows a geometric Brownian Motion which is orthogonal to the martingale part of X. To obtain their model we put

$$\sigma(t, x, v) := \sqrt{v},\ b(t, x, v) := \bar{b}v,\ \eta_1(t, x, v) := 0 \text{ and } \eta_2(t, x, v) := \delta v \tag{18}$$

for constants $\bar{b}$, δ with $\delta > 0$.

WIGGINS [71] assumes that the logarithm of σ_t^2 follows an arithmetic Ornstein-Uhlenbeck process. To obtain his model we put $\sigma(t, x, v) := \exp(\frac{1}{2}v)$, $b = \bar{v} - \kappa v$, $\eta_1(t, x, v) := 0$ and $\eta_2(t, x, v) := \delta$ for constants κ, $\bar{v}$ and δ with $\delta > 0$.

SCOTT [65] and STEIN and STEIN [69] assume that the state variable v follows the same arithmetic Ornstein-Uhlenbeck-process as in the model of

[9] Two (local) martingales are called orthogonal if their quadratic covariation process is zero.

Wiggins but they take $\sigma(t,x,v) := |v|$. If $\bar{v} = 0$ their model allows for an equivalent, more convenient description. We take $\sigma(t,x,v) := \sqrt{v}$ and model the dynamics of v by the familiar square root process introduced by COX, INGERSOLL, and ROSS [12] as a model for the short-term interest rate:

$$dv_t = (\delta^2 - 2\kappa v_t)dt + 2\delta\sqrt{v_t}\, dW_{2,t} \tag{19}$$

As shown by IKEDA and WATANABE [48] this SDE admits a unique strong solution which is nonnegative. It can be shown that the law of the two-dimensional process $(X_t, \sigma_t^2)_{0\leq t\leq\infty}$ is the same, no matter which of the two descriptions we use. Hence from an empirical viewpoint the two models are equivalent, as an observer is of course confined to recording the trajectories of X.[10] This clarifies a point raised by BALL and ROMA [3].

HESTON [43] also works with the specification $\sigma(t,x,v) = \sqrt{v}$ and models the dynamics of v by a square root process. In contrast to all the previous models he allows for nonzero covariation between X and v. As mentioned in Section 3 this is of empirical relevance as many financial time series exhibit significant negative correlation between returns and volatility innovations. Formally we obtain Heston's model by putting

$$a(t,x,v) := \lambda v,\ b(t,x,v) := \theta - \kappa v,\ \eta_1 := \rho\delta\sqrt{v},\ \eta_2 := \delta\sqrt{1-\rho^2}\,\sqrt{v} \tag{20}$$

for constants λ, θ, κ, ρ and δ with $\delta > 0$ and $\rho \in (-1,1)$.[11]

REMARK. HESTON [43] and in particular DUAN [19] suggest that — as in the case of the STEIN and STEIN [69] model — the model (20) is equivalent to a model where the dynamics of v are given by an arithmetic Ornstein-Uhlenbeck process which is driven by W_1 and W_2 and where $\sigma(t,x,v) := |v|$. However, if the covariation between X and v does not vanish, by computing the infinitesimal generator of (X,σ^2) it can be shown that the law of the process (X,σ^2) obtained in that way differs from the law this process obeys in the Heston model.

5.2. GARCH-Models as Diffusion Approximations

We now discuss the approximation of continuous-time SV-models by GARCH models which are set up in discrete time. This problem was first studied by NELSON [57]. Extensions of Nelson's results can be found in DUAN [19].

The approximation results we shall present here are of interest in the study of continuous-time SV-models for a number of reasons. To begin with such results are very helpful when it comes to estimating the parameters of SV-models.

[10] With continuous observations of the asset price process an observer can (theoretically) back out the path followed by σ_t^2 from the observed path of the stock price process using the quadratic variation of X along a suitable sequence of refining partitions of the time axis, see PROTTER [60], chapter 2.6. However, he is unable to distinguish between the different models for v. Of course in practice X can be observed only at discrete points in time, such that even the estimation of σ_t^2 poses a serious problem, see Section 5.2 below.

[11] The reason for assuming $\mu(t,x,v) = \lambda\sqrt{v}$, which contradicts (17) will become apparent in Section 6.2.

In practice we can observe the stock price process only at discrete points in time. We may now fit a discrete-time time series model such as a GARCH-model to our observations. If the time elapsing between the observations is "small" an approximation theorem gives some support to using the parameters of the discrete-time model in determining the parameters of the diffusion model. However, a word of warning is in order. While this procedure seems to work quite well, see e.g. the discussion in Ghysels, Harvey, and Renault [38], Section 4.3, it is unclear wether the estimates obtained in this way are actually unbiased. This is an important topic for research. For further information on the estimation of diffusion models from discrete observations see e.g. Dacunha-Castelle and Florens-Zmirou [14] or the survey articles Göng [39], Ait-Sahalia [1] and Ghysels, Harvey, and Renault [38].

Moreover some authors have recently developed option pricing models where the price process of the risky asset is given by a GARCH-type model, see e.g. Amin and Ng [2] or Duan [18]. Now the convergence of GARCH-models to continuous-time SV-models implies that the option prices obtained in these models are close to the option prices obtained in the limiting diffusion model, see also section 6.3 below. Hence we may use the results obtained in the discrete time framework to draw conclusions concerning the qualitative properties of option prices in certain continuous-time models. Finally there has always been some discussion if real-world asset prices are better described by discrete-time models or by continuous-time models. Approximation results are of interest here, as they may help to reconcile both approaches. For an in-depth discussion of results on weak convergence of asset price processes and their significance for derivative asset analysis see e.g. Duffie and Protter [21].

Assume that we are given a sequence of observations $(X_{t_k})_{k\in\mathbb{N}}$ of our stock price process at discrete, equidistant points in time $(t_k)_{k\in\mathbb{N}}$. Define the return process $(R_k)_{k\in\mathbb{N}}$ by $R_k = \ln X_{t_k} - \ln X_{t_{k-1}}$. All GARCH-type models considered in this paper assume the following dynamics for the sequence $(R_k)_{k\in\mathbb{N}}$

$$R_k = \frac{1}{2}\, h_{k-1} + \sqrt{h_{k-1}}(\varepsilon_k + \mu) \tag{21}$$

Here $(\varepsilon_k)_{k\in\mathbb{N}}$ is an i.i.d sequence of standardized random variables and μ is a constant. Hence h_{k-1} — which is supposed to be known at time t_k — equals the conditional variance of R_k given information up to time t_{k-1}.[12] The existing GARCH-models mainly differ in the specification of the dynamics imposed on the sequence $(h_k)_{k\in\mathbb{N}}$. An exhaustive survey of these models is given in Bollerslev, Chou, and Kroner [9]. Here we confine ourselves to introducing the models we need for our analysis of option pricing in stochastic volatility models. The first GARCH-model in the literature is the linear GARCH(1,1) (LGARCH)-model introduced by Bollerslev [8]. Here we have the following dynamics of h:

[12] Note the slightly different parametrization. Most authors denote the conditional variance of R_k by h_k which is taken to be predictable, whereas we denote this variance by h_{k-1} and assume the series $(h_k)_{k\in\mathbb{N}}$ to be merely adapted.

$$h_k = \beta_0 + \beta_1 h_{k-1} + \beta_2 h_{k-1}\varepsilon_k^2 \tag{22}$$

for positive constants β_0, β_1, β_2.

The following models have been developed because researchers wanted to incorporate the correlation between asset return and volatility innovations into their analysis. NELSON [58] proposed the EGARCH-model (exponential GARCH) where

$$\ln h_k = \beta_0 + \beta_1 \ln h_{k-1} + \beta_2 \left(|\varepsilon_k| - E[|\varepsilon_k|]\right) + \beta_3 \varepsilon_k \,. \tag{23}$$

Here the term $\beta_3\varepsilon_k$ takes account of the correlation between asset return and volatility. ENGLE and NG [29] used the following extension of the LGARCH-model, which is usually referred to as NGARCH-model,

$$h_k = \beta_0 + \beta_1 h_{k-1} + \beta_2 h_{k-1}(\varepsilon_k - c)^2 \,, \tag{24}$$

for positive constants β_0, β_1, β_2. For $c > 0$ ($c < 0$) returns and innovations of h_k are negatively (positively) correlated, for $c = 0$ we are back to the model (22).

We will now of certain GARCH-processes to a limiting diffusion process belonging to the class of SV-models. For each n consider a sequence of equidistant time points $0 = t_0^n < \ldots < t_k^n < \ldots$ and suppose that $\Delta^n := t_k^n - t_{k-1}^n$ tends to zero as $n \to \infty$. Suppose $(X^n)_{n\in\mathbb{N}}$ is a sequence of stock price processes where each process X^n is observed at the sequence $(t_k^n)_{k\in\mathbb{N}}$. Suppose further that every process X^n follows one of the previously introduced GARCH-models. In what follows we will identify a sequence $\xi_0^n, \xi_1^n, \ldots$ defined for times $t_0^n, t_1^n, \ldots$ with the RCLL (right continuous with left limits) function

$$\xi_t^n := \sum_{k=0}^{\infty} \xi_k^n 1_{\{t_k^n \le t < t_{k+1}^n\}} \tag{25}$$

This allows us to talk of convergence in distribution on the Skohorod space, see e.g ETHIER and Kurtz [30], chapter 3.

The results on the convergence of GARCH models we present here are due to NELSON [57] and DUAN [19]. Their results can be proved by applying ETHIER and KURTZ [30, Theorem 7.4.1]. The essence of this theorem can be summarized as follows: Suppose that the conditional mean and the conditional covariance of a given sequence of processes X^n converge after suitable rescaling to certain well behaved functions b and a on $\mathbb{R}^d$, and that the jumps of X^n converge to zero. Then the sequence X^n converges in distribution to the solution X of the SDE with drift $b(X_t)$ and with quadratic variation $a(X_t)dt$, provided that this equation admits a unique weak solution.

Let us now consider the previously introduced GARCH-models. Defining $Z_k^n := \ln X_k^n$ we get from equation (21) the following dynamics for Z_k^n

$$Z_k^n = Z_{k-1}^n + \frac{1}{2} \cdot h_{k-1}^n + \sqrt{h_{k-1}^n}(\sqrt{\Delta^n} \cdot \varepsilon_k^n + \mu^n) \,. \tag{26}$$

We make the following

ASSUMPTION 5.2. *As $n \to \infty$ we have $(\Delta^n)^{-1} \cdot \mu^n \to \mu$ for some constant μ. For every n $(\varepsilon_k^n)_{k\in\mathbb{N}}$ is an i.i.d sequence of random variables with variance equal to 1. The distribution of the ε_k^n is symmetric around the origin, has finite moments up to order 4 and — for simplicity — is independent of n.*

CONVERGENCE OF THE EGARCH-MODEL: If we define $v_k^n := \ln(h_k^n)$ the EGARCH specification implies the following form of dynamics for v^n:

$$v_k^n := v_{k-1}^n + \alpha_0^n + \alpha_1^n v_{k-1}^n + \alpha_2^n \left(|\varepsilon_k^n| - E[|\varepsilon_k^n|]\right) + \alpha_3^n |\varepsilon_k^n| \,, \tag{27}$$

where α_1^n corresponds to $\beta_1 - 1$ in (23). Now we may state

PROPOSITION 5.3. *Assume that as $n \to \infty$*

$$\frac{1}{\Delta^n}\alpha_0^n \to \alpha_0 \,,\ \frac{1}{\Delta^n}\alpha_1^n \to \alpha_1 \,,\ \frac{1}{\sqrt{\Delta^n}}\alpha_2^n \to \alpha_2 \ \text{and}\ \frac{1}{\sqrt{\Delta^n}}\alpha_3^n \to \alpha_3$$

for constants $\alpha_0, \ldots, \alpha_3$. Then the two-dimensional process (Z_t^n, v_t^n) obtained from (Z_k^n, v_k^n) via the identification (25) converges in distribution to the solution of the SDE

$$dZ_t \;=\; (\mu\sqrt{h_t} + \frac{1}{2}h_t)dt + \sqrt{h_t}dW_{1,t} \,, \tag{28}$$

$$dv_t \;=\; (\alpha_0 + \alpha_1 v_t)dt + \alpha_3 dW_{1,t} + \alpha_2\sqrt{\operatorname{var}(|\varepsilon_1^1|)}dW_{2,t} \,, \tag{29}$$

where h_t is shorthand for $\exp(v_t)$. Hence the EGARCH-model yields — under suitable rescaling — a diffusion approximation to the exponential Ornstein-Uhlenbeck model proposed for instance by [71].

REMARK. Suppose we are given parameter estimates $\beta_0, \ldots, \beta_3$ for the EGARCH model (23) obtained from discrete, equidistant observations of X, where Δt, the time between two observations is relatively small. We then define estimates for the diffusion model (28) (29) as follows.

$$\alpha_0 = \beta_0 \cdot (\Delta t)^{-1},\ \alpha_1 = (\beta_1 - 1) \cdot (\Delta t)^{-1},\ \alpha_2 = \beta_2 \cdot (\Delta t)^{-1/2},\ \alpha_3 = \beta_3 \cdot (\Delta t)^{-1/2} \,.$$

Proposition 5.3 applied to the sequence of EGARCH models with coefficients $\alpha_0^n = \alpha_0(\Delta^n), \ldots, \alpha_3^n = \alpha_3(\Delta^n)^{1/2}$ tells us that for Δt small our estimated EGARCH model and the diffusion model are close to each other in the sense of convergence in distribution which supports our choice for the parameters of the diffusion.

CONVERGENCE OF THE NGARCH-MODEL. Consider a sequence of NGARCH-models with return dynamics given by equation (26) and Assumption 5.2 and with dynamics of the conditional variance given by

$$h_k^n = \beta_0^n + \beta_1^n h_{k-1}^n + \beta_2^n h_{k-1}^n (\varepsilon_k^n - c^n)^2 \,. \tag{30}$$

To guess the form of a possible diffusion limit we decompose $(\varepsilon_k^n - c^n)^2$ into two uncorrelated random variables as follows

$$(\varepsilon_k^n - c^n)^2 = \left((\varepsilon_k^n)^2 - 1\right) + \left(-2c^n \varepsilon_k^n + (c^n)^2 + 1\right) ,$$

and introduce new coefficients $\alpha_0^n, \ldots, \alpha_3^n$ via

$$\alpha_0^n = \beta_0^n ,\ \alpha_1^n = \beta_1^n - 1 + \beta_2^n \left((c^n)^2 + 1\right) ,\ \alpha_2^n = \beta_2^n ,\ \alpha_3^n = \beta_2^n(-2c^n) . \quad (31)$$

Now (30) writes itself in the following form

$$h_k^n = h_{k-1}^n + \alpha_0^n + \alpha_1^n h_{k-1}^n + \alpha_2^n h_{k-1}^n \left((\varepsilon_k^n)^2 - 1\right) + \alpha_3^n h_{k-1}^n \varepsilon_k^n . \quad (32)$$

From this representation we get:

PROPOSITION 5.4. *Suppose that for $n \to \infty$*

$$\frac{1}{\Delta^n}\alpha_0^n \to \alpha_0 ,\ \frac{1}{\Delta^n}\alpha_1^n \to \alpha_1 ,\ \frac{1}{\sqrt{\Delta^n}}\alpha_2^n \to \alpha_2 \ \textit{and}\ \frac{1}{\sqrt{\Delta^n}}\alpha_3^n \to \alpha_3$$

for constants $\alpha_0, \ldots, \alpha_3$. Then the two-dimensional process (Z_t^n, v_t^n) obtained from (Z_k^n, v_k^n) via the identification (25) converges in distribution to the solution of the SDE

$$dZ_t = (\mu\sqrt{h_t} + \frac{1}{2}h_t)dt + \sqrt{h_t}dW_{1,t} , \quad (33)$$

$$dh_t = (\alpha_0 + \alpha_1 h_t)dt + \alpha_3 h_t dW_{1,t} + \alpha_2 \sqrt{\operatorname{var}((\varepsilon_1^1)^2)}\, h_t dW_{2,t} . \quad (34)$$

The proposition shows that the NGARCH yields a diffusion approximation to an (extended) Hull-White model, possibly with nonzero covariation between stock price process and state variable. To obtain an estimate for the coefficients of (34) from an estimate $\beta_0, \beta_1, \beta_2, c$ of the parameters of the NGARCH-model we may proceed as in case of the EGARCH-model. Of course in this case we must use (31) when defining the parameters $\alpha_0, \ldots, \alpha_3$ of the limiting diffusion model. Note in particular that the sign of the covariation between stock price process and state variable is entirely determined by the sign of the estimated parameter c.

6. PRICING AND HEDGING OF DERIVATIVES IN SV-MODELS

6.1. Approaches to Derivative Pricing under Incompleteness

While SV-models do a better job in fitting the behaviour of actual (stock)-market data than DV-models, this increase in realism comes at a cost. SV-models are incomplete, that is there are derivative assets that cannot be replicated by dynamic trading in stock and bond. As explained in Section 2 this is equivalent to the fact that there are now many probability measures $Q \sim P$ such that the stock price process is a (local) Q-martingale.

The Set of Equivalent Martingale Measures The next proposition characterizes the set of all equivalent local martingale measures for the stock price process defined in Assumption 5.1. For similar results see e.g. HOFFMANN, PLATEN and SCHWEIZER [46] and the references given therein.

PROPOSITION 6.1. *1. Under Assumption 5.1 a probability measure Q that is equivalent to P on $\mathcal{F}_T$ is a local martingale measure for X on $\mathcal{F}_T$ if and only if there is a progressively measurable process $\nu = (\nu_t)_{0\le t\le T}$ with $\int_0^T \nu_s^2 ds < \infty$ P a.s. such that the following holds: The local martingale $(G_t)_{0\le t\le T}$ with*

$$\begin{aligned} G_t \quad := \quad & \exp\Big(\int_0^t -\mu(s, X_s, v_s) dW_{1,s} \\ & + \int_0^t \nu_s dW_{2,s} - \tfrac{1}{2}\int_0^t \mu^2(s, X_s, v_s) + \nu_s^2 \, ds\Big) \end{aligned} \tag{35}$$

satisfies $E[G_T] = 1$ and we have $G_T = dQ/dP$ on $\mathcal{F}_T$.

2. Suppose that Q is an equivalent local martingale measure corresponding to some process ν. Then X and v solve the following SDE under Q

$$dX_t = \sigma(t, X_t, v_t) X_t dW_{1,t}^Q \tag{36}$$

$$\begin{aligned} dv_t \quad = \quad & [b(t, X_t, v_t) - \eta_1(t, X_t, v_t)\mu(t, X_t, v_t) + \eta_2(t, X_t, v_t)\nu_t]\, dt \\ & + \eta_1(t, X_t, v_t) dW_{1,t}^Q + \eta_2(t, X_t, v_t) dW_{2,t}^Q \,. \end{aligned} \tag{37}$$

The proof is given in Appendix A.

REMARK. SIN [67, 68] gives conditions for the solution in (36) to be actually a martingale (and not only a local martingale). He shows that for the examples introduced in Section 5.1 the martingale property of X is equivalent to the covariation between X and v being nonpositive.

In the Finance literature the process ν is usually referred to as *market price of volatility risk process.* Proposition 6.1 shows that there is a one to one correspondance between market price of volatility risk processes ν satisfying some regularity conditions and equivalent (local) martingale measures. In particular market incompleteness is equivalent to nonuniqueness of the market price of risk process.

Let us now turn to the pricing and hedging of derivatives. Here a conceptual problem arises: how should we value a contingent claim $\bar{H}$ for which a replicating portfolio does not exist? From the viewpoint of arbitrage pricing theory any price process $H = (H_t)_{0\le t\le T}$ with $H_T = \bar{H}$ is in order, provided that the two-dimensional price system (X, H) precludes arbitrage opportunities. Following the fundamental paper by DELBAEN and SCHACHERMAYER [15] we are "on the safe side" if the process (X, H) admits an equivalent local martingale measure. Hence every price process of the form $H_t = E^Q[\bar{H}|\mathcal{F}_t]$ where $Q \sim P$ and X is a local Q-martingale is acceptable, provided of course that $\bar{H}$ is Q-integrable. We will therefore call every conditional expectation $E^Q[[X_T - K]^+|\mathcal{F}_t]$ an *option value.*

The next proposition shows that for certain SV-models only very elementary bounds on the range of option values can be given. Consider within the framework of assumption 5.1 the Hull-White model (18). Denote by $\mathcal{Q}$ the set of all equivalent local martingale measures for X. Then we have the following result on the range of option values for a European call option with exercice price $K \ge 0$:

PROPOSITION 6.2. *In the Hull-White model (18) we have for any* $0 \le t < T$

$$\sup_{Q\in\mathcal{Q}} E^Q\left[[X_T - K]^+|\mathcal{F}_t\right] = X_t \text{ and } \inf_{Q\in\mathcal{Q}} E^Q\left[[X_T - K]^+|\mathcal{F}_t\right] = [X_t - K]^+ .$$

The proof is given in Appendix B. I have chosen to state this result in the framework of the Hull-White model (18), because the corresponding SDE has an explicit solution which allows for an easy proof. A similar statement should hold in most SV-models where the stock price volatility is an unbounded process.[13] However, at least to my knowledge, a formal proof has not yet been given and even the above proposition is new. Related results on the range of option prices in a model where $\ln X$ follows a Levy-process with unbounded jumps have been obtained by EBERLEIN and JACOD [24].

Obviously in an arbitrage-free market a call option is always worth less than the underlying security. On the other hand we know from Merton's theorem on the equivalence of European and American options that the price of a European call on some non-dividend paying asset must exceed the intrinsic value $[X_t - K]^+$. Proposition 6.2 now tells us that — at least in case of the model (18) — these elementary bounds are the sharpest possible bounds for the range of option prices consistent with absence of arbitrage.

In light of Proposition 6.2 we need additional arguments to arrive at a well determined price for options. Many articles in the Finance literature simply choose one particular market price of risk process and justify their choice by — often rather loose — economic equilibrium arguments. For instance it is often argued that "volatility risk can be diversified away" which is used as a rationale for simply taking $\nu_t \equiv 0$. This approach is taken in the work mentioned in section 5.1.

The literature on option pricing in a GARCH framework proceeds similarly, see DUAN [18] or AMIN and NG [2]. In these models option prices are defined as expected value of the terminal payoff; expectations are taken in a transformed GARCH model where the return process is given by (21) but with $\mu = 0$ — hence the asset price process forms a martingale — and where the conditional variance follows one of the models introduced in Section 5.2. Again there are many equivalent martingale measures and equilibrium arguments are used to justify the choice of a particular "pricing measure".

These approaches are not very satisfactory as the arguments justifying the choice of a particular "pricing measure" are often somewhat ad-hoc. Moreover, the risk-management of derivatives, is not adressed in this literature. As this topic is of great importance to practioners we will now discuss two approaches to derivative asset analysis in incomplete markets which are based on hedging arguments.

Superreplication If the precise duplication of a contingent claim is not feasible one might try to find a *superreplicating strategy*, i.e. the "cheapest" selffinancing strategy with terminal value no smaller than the payoff of the contingent

[13] This conjecture is confirmed by ongoing work of C. Sin and the author.

claim. This concept is developed by El Karoui and Quenez [27]. To explain the results of this paper we have to introduce some formal definitions. Remember the definition of the cost process C in (3).

Definition 6.3. *Consider a contingent claim $\bar{H}$. An adapted RCLL process H with $H_T = \bar{H}$ is called an admissible price for sellers, if H is the value process of some trading strategy (ξ, η) with nonincreasing cost process. An admissible price process for sellers H^* will be called the ask price for the contingent claim $\bar{H}$, if we have for any other admissible price process for sellers H and for all $t \in [0, T]$ the inequality $H_t^* \leq H_t$ P a.s..*

This definition deserves a comment. Suppose that an investor sells at time $t < T$ the claim $\bar{H}$ at an admissible selling price H_t. By following the corresponding portfolio strategy he can then completely eliminate the risk incurred by selling the claim and moreover he earns the nonnegative amount $-(C_T - C_t)$. Hence he will certainly agree to sell the claim for the price H_t. The following is an example for an admissible price process for sellers in the case of a European call option. Define

$$H_t = X_t \,,\ \xi_t = 1 \ \text{ for } 0 \leq t < T \ \text{ and } H_T = [X_T - K]^+ \,,\ \xi_T = 0\,. \tag{38}$$

The cost process is then given by $C_t = 0$ for $t < T$ and $C_T = [X_T - K]^+ - X_T$.

Note that it is not clear that an ask-price for a contingent claim exists. At least for nonnegative claims combining El Karoui and Quenez [27], Theorem 2.1.1 and Theorem 2.2.1) yields the following remarkable result.

Theorem 6.4. *Under Assumption 5.1 the ask price process H^* exists for every contingent claim $\bar{H}$ whose payoff is bounded below and satisfies $\sup_{Q \in \mathcal{Q}} E^Q[\bar{H}] < \infty$. It is given by*

$$H_t^* = \sup_{Q \in \mathcal{Q}} E^Q[\bar{H} | \mathcal{F}_t]\,.$$

The crucial point of this theorem is the fact that that the process $\sup_{Q \in \mathcal{Q}} E^Q [\bar{H} | \mathcal{F}_t]$, which is a natural lower bound for every admissible price process for sellers, can be represented as sum of a stochastic integral with respect to X and a nonincreasing process. Theorem 6.4 holds in very general setups. El Karoui and Quenez [27] prove it for a general diffusion model; for an extension to general semimartingales see Kramkov [53].

At a first glance superreplication seems to be a very attractive concept for the pricing and the hedging of derivatives in incomplete markets (in particular from the viewpoint of risk-management of written derivative contracts). Unfortunately in our SV-framework it may lead to answers which are not very satisfactory. Remember Proposition 6.2. There we showed that for some typical SV-model $\sup\{E^Q[[X_T - K]^+ | \mathcal{F}_t]\,,\ Q \in \mathcal{Q}\} = X_t$. As a call option satisfies the hypothesis of Theorem 6.4 the ask price process and the corresponding hedge portfolio are given by (38); in other words the superreplicating strategy for a call option is to buy the stock.

Note however, that the idea of superreplication may lead to interesting results on pricing and hedging derivatives if a priori bounds for the stock price volatility are known, i.e. if we know that a.s. $\sigma(t, X_t, v_t) \leq \bar{\sigma}$ for some constant $\bar{\sigma}$ for all t. In that case the Black-Scholes price for the upper volatility bound is an admissible price process for sellers; the portfolio strategy is given by the Black-Scholes strategy corresponding to $\bar{\sigma}$. For a proof and extensions of this result see the interesting paper ELKAROUI, JEANBLANC-PICQUÉ and SHREVE [26].

(Local) Risk-Minimization Even in an incomplete market a part of the risk incurred by selling derivatives can be hedged by dynamic trading in the underlying asset. In the theory of (local) risk-minimization which has been developed in the papers FÖLLMER and SONDERMANN [34] SCHWEIZER [64] and FÖLLMER and SCHWEIZER [33], one seeks to find a trading strategy that reduces the actual risk of a derivative position to some "intrinsic component." While the computation of the strategy usually involves the computation of "prices" for contingent claims, the focus of this theory is not on the valuation of derivatives but on the reduction of risk.

Let us now explain this approach in more detail. Recall for a trading strategy (ξ, η) the definition of the cost-process C in (3) and assume that value process and cost process are square integrable. In the theory of local risk-minimization the conditional variance of C under the "real-world" probability measure P is used as a measure for the risk of a strategy. For a given claim $\bar{H}$ one tries to determine a strategy (ξ^*, η^*) with terminal value equal to $\bar{H}$ that minimizes at each time t the remaining risk

$$R_t := E^P[(C_T - C_t)^2 | \mathcal{F}_t]. \tag{39}$$

Here the minimization is over all admissible continuations of (ξ^*, η^*) after t with terminal value equal to $\bar{H}$.

FÖLLMER and SONDERMANN [34] have studied existence and uniqueness of such a strategy if the stock price process is a P-martingale. In that case a unique risk-minimizing strategy exists. It can be computed by means of the well known Kunita-Watanabe decomposition[14] of the P-martingale $H_t = E^P[\bar{H} | \mathcal{F}_t]$ with respect to the P-martingale X.

Let us now turn to the general situation where X is only a semimartingale under P. As shown by SCHWEIZER [64] in that situation a globally risk-minimizing strategy need not exist. He therefore introduces a criterion of local risk-minimization. Roughly speaking a strategy (ξ^*, η^*) is locally risk-minimizing if it minimizes the remaining risk over all strategies that "deviate" from (ξ^*, η^*) only over a sufficiently short time period. SCHWEIZER [64] shows that under some technical conditions[15] a strategy is locally risk-minimizing if

[14] see e.g. KARATZAS and SHREVE [51], Proposition 3.4.14

[15] Besides certain integrability conditions he assumes that the finite variation part in the semimartingale decomposition of X is a continuous process. This condition is satisfied for all continuous semimartingales and hence in particular for our SV-models.

and only if the associated cost process is a martingale orthogonal to the martingale part of X. To compute such a strategy we have to find a decomposition of our claim $\bar{H}$ of the following form

$$\bar{H} = H_0 + \int_0^T \xi_s^H dX_s + L_T^H \, , \tag{40}$$

where L^H is a P-martingale orthogonal to the martingale part of Xunder P. Given such a decomposition we may define a locally risk-minimizing strategy by putting $\xi^* := \xi^H$ and $C := L$. In particular the strategy is still *mean-selffinancing*, i.e. the cost process is a P-martingale and $E^P[C_T] = 0$. Note that in the case where X is a P-martingale the decomposition (40) reduces to the Kunita-Watanabe decomposition of the P-martingale H with respect to X. If X is only a semimartingale the decomposition (40) is usually referred to as *Föllmer-Schweizer decomposition*.

In the case where X is a semimartingale with continuous sample paths — hence in particular in our SV-models — Föllmer and Schweizer [33] have proposed the following approach to computing the decomposition (40). As a first step one has to determine the *minimal martingale measure* Q^*. It is characterized by the following property:

> X is a Q^* martingale and every P-martingale that is orthogonal to the martingale part of X under P remains a martingale under Q^*.

Föllmer and Schweizer [33] show that for a contingent claim $\bar{H}$ which is Q^*-integrable the decomposition (40) is uniquely determined. It exists under some integrability assumptions and is then given by the Kunita-Watanabe decomposition of the Q^* martingale $H_t = E^{Q^*}[\bar{H}|\mathcal{F}_t]$, $0 \leq t \leq T$ with respect to the Q^*-martingale X. To compute the decomposition (40) one can therefore compute this Kunita-Watanabe decomposition under Q^* and check the integrability conditions.

Let us now turn to the application of this recipe in the context of our SV-models. The minimal martingale measure Q^* is the martingale measure corresponding to a market price of volatility risk process $\nu \equiv 0$. This follows either from the property characterizing Q^* or from the formula for the density dQ^*/dP given by Föllmer and Schweizer [33]. Consider a European call option. We get from the Markov property of the process (X, v) under Q^*

$$E^{Q^*}[[X_T - K]^+|\mathcal{F}_t] = E^{Q^*}_{(X_t,v_t)}[[X_{T-t} - K]^+] =: g(t, X_t, v_t) \, . \tag{41}$$

Under some regularity conditions on the coefficients of the diffusion the function g is smooth. In that case we have

Proposition 6.5. *Suppose that (X, v) satisfy Assumption 5.1 and that the function g defined in (41) is of class $C^{1,2}([0,T] \times \mathbb{R}^+)$. Then the local risk minimizing hedge strategy (ξ^*, η^*) for a European call option is given by*

$$\xi_t^* = \frac{\partial}{\partial x} g(t, X_t, v_t) + \frac{\eta_1(t, X_t, v_t) \frac{\partial}{\partial v} g(t, X_t, v_t)}{\sigma(t, X_t, v_t) X_t} \quad \textit{and } \eta_t^* = g(t, X_t, v_t) - \xi_t^* X_t \tag{42}$$

In particular the value process of this strategy is given by $V_t^* := g(t, X_t, v_t)$.

REMARK. Note that classical Δ-hedging where $\xi_t = \frac{\partial}{\partial x} g(t, X_t, v_t)$ is not optimal in the sense of local risk-minimization whenever the covariation between X and v is different from zero. (42) also points to a minor error in FÖLLMER, PLATEN, and SCHWEIZER[46]. These authors claim that the locally risk-minimizing hedge-portfolio in a SV-model is always given by $\xi_t = \frac{\partial}{\partial x} g(t, X_t, v_t)$ and $\eta_t = g(t, X_t, v_t) - \xi_t^* X_t$, see equations (2.5) and (2.6) of their paper. As shown above this is wrong whenever $\eta_1 \neq 0$.

PROOF. As g is of class $C^{1,2}([0,T] \times \mathbb{R}^+)$ we may use Itô's formula to obtain the dynamics of $g(t, X_t, v_t)$ under Q^*. Note that the finite variation terms must cancel as $g(t, X_t, v_t)$ is a Q^*-martingale.[16] Using the SDE solved by (X, v) under Q^* (see Proposition 6.1) yields

$$\begin{aligned}
[X_T - K]^+ &= g(T, X_T, v_T) \\
&= g(0, X_0, v_0) + \int_0^T \frac{\partial}{\partial x} g(t, X_t, v_t) \sigma(t, X_t, v_t) X_t dW_{1,t}^{Q^*} \\
&+ \int_0^T \frac{\partial}{\partial v} g(t, X_t, v_t) \eta_1(t, X_t, v_t) dW_{1,t}^{Q^*} \\
&+ \int_0^T \frac{\partial}{\partial v} g(t, X_t, v_t) \eta_2(t, X_t, v_t) dW_{2,t}^{Q^*} \\
&= g(0, X_0, v_0) + \int_0^T \xi_t^* dX_t \\
&+ \int_0^T \frac{\partial}{\partial v} g(t, X_t, v_t) \eta_2(t, X_t, v_t) dW_{2,t}^{Q^*} .
\end{aligned}$$

We now define the Q^*-martingale L by $L_t := \int_0^t \frac{\partial}{\partial v} g(s, X_s, v_s) \eta_2(s, X_s, v_s) dW_{2,s}^{Q^*}$. As $W_1^{Q^*}$ and $W_2^{Q^*}$ are orthogonal, L is orthogonal to $\int_0^t \xi_s^* dX_s$. This shows that we have found the Kunita-Watanabe decomposition of the Q^*-martingale $g(t, X_t, v_t)$ with respect to the Q^*-martingale X and hence the Föllmer-Schweizer decomposition of our call option. □

REMARK. Note that we have identified the cost process of our locally risk-minimizing hedging-strategy in the proof of the above proposition. It is given by

$$C_t := \int_0^t \frac{\partial}{\partial v} g(s, X_s, v_s) \eta_2(s, X_s, v_s) dW_{2,s}^{Q^*} \tag{43}$$

[16] This yields a parabolic PDE for g which can be used to compute g numerically, see Section 6.2 below.

As the market price of volatility risk process corresponding to Q^* is given by $\nu \equiv 0$ we have the equality $W_2^{Q^*} = W_2$; this shows again that C is both a P-martingale and a Q^*-martingale.

Often the minimal martingale measure is used for the pricing of contingent claims. This implies that one associates a price of zero to the claim with payoff C_T. As this claim has zero expected value under P, using Q^* for the pricing of contingent claims implies that the seller has to bear the whole intrinsic risk of the claim without receiving any compensation for it. When selling derivatives to clients a market maker could charge the Q^*-price plus some markup which might for instance be proportional to the variance of the cost process. More generally one could use principles from insurance mathematics to determine a price for the totally unhedgeable claim C_T; see [28] for a stimulating discussion on the interplay of actuarial and financial pricing principles. The minimal martingale measure could be used for the internal valuation and the risk-measurement of a book of derivative assets.

In Section 6.3 we compile some evidence on the qualitative properties of option prices in SV-models. It turns out that these option prices exhibit the same qualitative behaviour as market prices for options. Since option traders usually correct for the known deficits of the classical Black-Scholes model when quoting their prices, this gives some hope that the concept of local risk-minimization applied in the framework of SV-models could be a valuable tool for improving the risk-management of derivatives. Clearly this is an important issue for further testing and research.

6.2. Computation of Option Values

As every approach to pricing and hedging options in SV-models involves the computations of option values, we will now survey certain analytical and numerical approaches to computing these conditional expectations. By the Markov property of our basic SV-model outlined in Assumption 5.1 it is enough to consider the computation of expected values $E^Q\left[[X_t - K]^+\right]$ where Q is an equivalent local martingale measure for X.

Analytical Approaches When looking for an analytical solution to this problem we have to distinguish two cases. First assume that the martingale parts of volatility process and asset price process are orthogonal and that drift and dispersion coefficient of the state variable do not depend on X. In that case the distribution of X_T conditional on the path followed by v is lognormal and we get

$$E^Q[(X_T - K)^+] = \int_{\mathbb{R}^+} C_{BS}(0, X_0, \bar{\sigma})\, \lambda^Q(d\bar{\sigma})\,. \tag{44}$$

Here $C_{BS}(0, X_0, \bar{\sigma})$ is the Black-Scholes price of the option as given by (9) and (10) and λ^Q denotes the distribution of the average variance $\bar{\sigma} = 1/T \cdot \int_0^T v_s ds$ under Q. To compute the expectation (44) one hence has to identify the distribution λ^Q. In the literature on option pricing under stochastic volatility several techniques have been proposed for this. Most of the papers concentrate on computing the moment generating function of λ^Q. As these approaches are reviewed in detail by [3] we will not treat them here.

The only contribution that deals with the computation of option values in a model where X and v are correlated is [43], who works in the model (20). We now give a slightly simplified version of his derivation of option prices, as this allows us to review certain arguments that are used over and over in modern continuous-time derivative asset analysis. We assume that (X, v) follow under Q the SDE

$$\begin{aligned} dX_t &= \sqrt{v_t} X_t dW_{1,t} && (45) \\ dv_t &= (\theta - \tilde{\kappa} v_t)dt + \delta\sqrt{v_t}(\rho dW_{1,t} + \sqrt{1-\rho^2} dW_{2,t}) && (46) \end{aligned}$$

for constants $\theta, \tilde{\kappa}, \delta > 0$ and $\rho \in (-1, 1)$. We moreover assume that $\theta \geq \delta^2/2$. As shown by [68], under this assumption X is a martingale and v is strictly positive. Note that it follows from Proposition 6.1, that (X, v) solve the above SDE if we consider the model (20) under the minimal martingale measure.[17] We have

$$E^Q[[X_T - K]^+] = E^Q[X_T \cdot 1_{\{\ln X_T > \ln K\}}] - KE^Q[1_{\{\ln X_T > \ln K\}}] \quad (47)$$

Since X is a strictly positive Q-martingale we have $E^Q[X_T] = X_0$. Hence we may define a new probability measure[18] Q^X by putting $dQ^X/dQ := X_T/X_0$ and get

$$E^Q[X_T \cdot 1_{\{\ln X_T > \ln K\}}] = X_0 E^{Q^X}[1_{\{\ln X_T > \ln K\}}] \,.$$

While the "exercice probabilities" in (47) cannot be computed explicitely, it is possible to give an analytic expression for the characteristic functions ϕ_1 and ϕ_2 of the distribution of $\ln X_T$ under Q^X and Q. By definition we have

$$\phi_1(\lambda) = E^{Q^X}[\exp(i\lambda \ln X_T)] \text{ and } \phi_2(\lambda) = E^Q[\exp(i\lambda \ln X_T)] \,.$$

We deal only with ϕ_1. As X_T is given by $X_T = X_0 \exp(\int_0^T \sqrt{v_s} dW_{1,s} - \frac{1}{2}\int_0^T v_s ds)$, Girsanov's theorem yields the following dynamics for the process $(\ln X, v)$ under Q^X.

$$\begin{aligned} d\ln X_t &= \sqrt{v_t}(dW_{1,t}^X + \sqrt{v_t}dt) - \frac{1}{2}v_t dt = \sqrt{v_t}dW_{1,t}^X + \frac{1}{2}v_t dt \\ dv_t &= (\theta - \tilde{\kappa} v_t) + \delta\rho v_t dt + \delta\sqrt{v_t}(\rho dW_{1,t}^X + \sqrt{1-\rho^2} dW_{2,t}), \end{aligned}$$

where $W_{1,t}^X := W_{1,t} - \int_0^t \sqrt{v_s} ds$ is a Q^X-Brownian motion. By the Markov property the conditional expectation $E^{Q^X}[\exp(i\lambda \ln X_T) | \mathcal{F}_t]$ is given by some

[17] Essentially this is the rationale behind our choice of the risk premium μ in (20).

[18] For a systematic analysis of the role of the measure Q^X from the viewpoint of the change of numeraire theory see [25].

function $g_\lambda(t, \ln X_t, v_t)$; obviously $\phi_1(\lambda) = g_\lambda(0, \ln X_0, v_0)$. Applying Itô's formula we see that the process $g_\lambda(t, \ln X_t, v_t)$ can be written as sum of stochastic integrals with respect to the Brownian motions W_1^X and W_2 and finite variation terms. As the process $g_\lambda(t, \ln X_t, v_t)$ is a martingale by definition, the finite variation terms must cancel. This yields the following PDE for g_λ:

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} g_\lambda(t,y,v) + \frac{1}{2} v \frac{\partial}{\partial y} g_\lambda(t,y,v) \\ &+ (\theta + (\delta\rho - \tilde{\kappa})v) \frac{\partial}{\partial v} g_\lambda(t,y,v) + \frac{1}{2} v \frac{\partial^2}{\partial y^2} g_\lambda(t,y,v) \\ &+ \rho\delta v \frac{\partial^2}{\partial y \partial v} g_\lambda(t,y,v) \\ &+ \frac{1}{2}\delta^2 v \frac{\partial^2}{\partial v^2} g_\lambda(t,y,v)\,,\ g_\lambda(T,y,v) = \exp(i\lambda y)\,. \end{aligned} \tag{48}$$

Guided by the form of the solution of the bond price equation in the term structure model of Cox, Ingersoll and Ross [12], Heston "guesses" a solution of the form

$$g_\lambda(t,y,v) = \exp(C(T-t) + D(T-t)v + i\lambda x)$$

for functions $C, D : [0,T] \to \mathbb{R}$ with $C(0) = D(0) = 0$. Substituting this candidate solution into the above PDE yields ordinary differential equations for C and D which are solved explicitely in Heston [43]. The option value can now be computed by inverting the characteristic functions ϕ_1 and ϕ_2 and evaluating the exercice probabilities; see again Heston [43] for details.

Numerical Approaches The numerical techniques used for the computation of option values belong to two groups. On the one hand researchers have used a Monte-Carlo approach combined with discretization schemes for the SDE (15), (16) to compute the option value. Monte-Carlo simulation is a well known tool in option pricing; a general survey of modern developments is Boyle, Broadie and Glasserman [10]. Techniques for the discretization and numerical solution of SDEs can be found in the book Kloeden and Platen [52]; for an application of these techniques in the context of SV-models see Hofmann, Platen, and Schweizer [46]. Monte Carlo simulations are also always used for the computations of option values in GARCH-models.

Alternatively researchers have noticed that — at least under some regularity conditions — in Markovian models the option value can be characterized by a parabolic PDE.[19] Usually *finite difference methods* are used for solving this PDE numerically. See Duffie [20],chapter 10 H, for an introduction to this technique and the book Willmott, Dewynne, and Howison [72] for an extensive treatment and applications to option pricing. Note that the "pricing PDE" contains two state variables, namely x and v, such that certain

[19] This PDE can be derived by an analogous argument as it is used in the derivation of the PDE (48).

simple methods which are designed particularly for equations with only one state variable cannot be applied to SV-models. As to a comparison of the two approaches, according to DUFFIE [20],p. 207, it can be said that "for problems involving one or two state variables it is typically the case that the PDE-approach requires fewer computations than the Monte Carlo approach to achieve the same accuracy", whereas for higher dimensional problems the Monte Carlo approach seems preferable.

6.3. Qualitative Properties of Option Values

We now collect evidence on the qualitative behaviour of option prices both in stochastic volatility models and in a GARCH framework. The convergence in distribution of GARCH models to continuous time SV models implies that option values obtained in GARCH models converge to the option values one would obtain in the limiting SV model.[20] Hence the qualitative behaviour of option values in both classes of models is the same. Therefore we will not distinguish between these types of models in this Section.

Generally speaking it can be said that the qualitative properties of option values predicted by the SV-models are close to the qualitative properties of observed option prices. In all SV-models we observe the smile pattern of implied volatility, i.e. increasing the volatility of the volatility leads to rising implied volatilities of in the money options and out of the money options whereas the prices of at the money options remain (roughly) unchanged. In case that volatility innovations and asset returns are uncorrelated there is even a formal proof of this observation which is due to RENAULT and TOUZI [61]. This interesting paper also compares hedge ratios in the Black-Scholes model to hedge ratios in SV-models. The authors find that "the usual hedging methods, through the Black-Scholes model, lead to an underhedged (resp. overhedged) position for in-the-money (resp. out of the money) options and a perfect partially hedged position for at the money options." HESTON [43] shows that SV-models can explain the skew pattern of implied volatility. If the covariation between X and v is negative — remember that this is the empirically relevant case — the left tail of the return distribution is spread out. Hence put options with a relatively low strike price rise in price. Option prices in SV-models seem to exhibit term structure effect: the implied volatility of options with short time to maturity reacts much stronger to changes in the current stock price volatility than does the implied volatility of options with a relatively long time to maturity. Again this behaviour is typical for the implied volatility of traded option contracts.

DUAN [17] carries out an analysis similar in spirit to the IDV models of section 4. He determines the parameters of an NGARCH-model by minimizing

[20] The convergence of the models must of course take place under the martingale measures Q^n and Q used for the computation of option values. The convergence of the values of put options, whose payoff is bounded, follows directly from the definition of convergence in distribution; the convergence of call values is implied by the put-call-parity and the convergence $E^{Q^n}[X_T^n] \to E^Q[X_T]$ which in turn follows from the martingale property of X^n and X and the assumed convergence of the initial values, $X_0^n \to X_0$.

the distance between the option values predicted by the NGARCH model and an observed implied volatility smile. He obtains a very good fit. More importantly, the parameter values he obtains have the same order of magnitude than the parameters one usually obtains by fitting an NGARCH model directly to the time series of the underlying asset price. In contrast to the IDV-models the parameter values obtained by Duan are relatively stable over time. By this we mean that the option prices predicted by a NGARCH model that has been calibrated to an implied volatility smile prevailing one week before fit the current volatility smile reasonably well.

Duan's paper is only a first study and should therefore not be taken as a rationale for claiming that SV-models are a better risk-management tool than IDV-models. Clearly more testing of both models is called for before a definitive statement of this type can be made. Also some care should be taking in saying that "stochastic volatility is the reason why we observe volatility smiles" or "the correlation between stock price volatility and asset returns causes the skew" as there are other possible explanations such as jumps, transaction costs, liquidity problems or even feedback effects from dynamic hedging, see Ghysels, Harvey, and Renault [38, section 2.2] and Platen and Schweizer [59]. Nonetheless the evidence compiled above suggests that SV-models are a good description of financial markets and might therefore help financial institutions to deal with the volatility risk of derivative contracts in a reasonable and consistent manner.

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Appendix

A. Proof of Proposition 6.1

Suppose that Q is an equivalent local martingale measure on $\mathcal{F}_T$ and denote by G the density martingale $G_t = dQ/dP|_{\mathcal{F}_t}$. By the martingale representation theorem we know that G can be written as stochastic integral

$$G_t = \int_0^t \xi_{1,s} dW_{1,s} + \int_0^t \xi_{2,s} dW_{2,s}$$

for progressively measurable integrands ξ_1 and ξ_2. As Q and P are equivalent G is strictly positive such that $\ln G$ is well defined. We get from Itô's formula

$$\ln G_t = \ln G_0 + \int_0^t \frac{\xi_{1,s}}{G_s} dW_{1,s} + \int_0^t \frac{\xi_{2,s}}{G_s} dW_{2,s} + \int_0^t \left(\frac{\xi_{1,s}}{G_s}\right)^2 + \left(\frac{\xi_{2,s}}{G_s}\right)^2 ds.$$

Hence G is of the form (35) with $\nu_t = \xi_{2,t}/G_t$. It remains to show that $\mu(t, X_t, v_t) = -\xi_{1,t}/G_t$. Now we obtain from Girsanov's theorem that X solves under Q the SDE

$$dX_t = \sigma(t, X_t, v_t) X_t \left(dW_{1,t}^Q + \left(\mu(t, X_t, v_t) + \frac{\xi_{1,t}}{G_t} \right) dt \right) .$$

Hence X is a Q-local martingale if and only if $\mu(t, X_t, v_t) = -\xi_{1,t}/G_t$.

Conversely, define for ν such that $E[G_T] = 1$ the measure Q by $dQ/dP|_{\mathcal{F}_T} = G_T$. Now it follows immediately from Girsanov's theorem that (X, v) solves the SDE (36), (37) under Q; hence X is a local Q-martingale. □

B. Proof of Proposition 6.2

We will denote for $\nu \in \mathbb{R}$ the equivalent martingale measure belonging to the constant market price of risk process $\nu_t \equiv \nu$ by Q^ν.[21] Obviously it is enough to show that

$$\sup_{\nu \in \mathbb{R}} E^{Q^\nu}[[X_T - K]^+ | \mathcal{F}_t] = X_t \text{ and } \inf_{\nu \in \mathbb{R}} E^{Q^\nu}[[X_T - K]^+ | \mathcal{F}_t] = [X_t - K]^+ .$$

By the Markov property of (X, v) under Q^ν it is enough to consider the case $t = 0$. As the covariation between X and v vanishes the distribution of X_T conditional on the path followed by v is lognormal, and we get

$$E^{Q^\nu}[(X_T - K)^+] = \int_{\mathbb{R}+} C_{BS}(0, X_0, \bar{\sigma}) \, \lambda^\nu(d\bar{\sigma}) . \tag{49}$$

Here $C_{BS}(0, X_0, \bar{\sigma})$ is the Black-Scholes price of the option as given by (9) and (10) and λ^ν denotes the distribution of the average variance $\bar{\sigma} = 1/T \cdot \int_0^T v_s ds$ under Q^ν. Inspection of the definition of C_{BS} immediately yields that for all $x > 0$

$$\lim_{\bar{\sigma} \to \infty} C_{BS}(0, x, \bar{\sigma}) = x \quad \text{and} \quad \lim_{\bar{\sigma} \to 0} C_{BS}(0, x, \bar{\sigma}) = [x - K]^+ . \tag{50}$$

Now the process v solves under Q^ν the SDE $dv_t = v_t \delta \nu dt + v_t \delta dW_{2,t}^{Q^\nu}$. Hence it equals

$$v_t = v_0 \exp(\delta \nu t) \cdot \exp(\delta W_{2,t}^{Q^\nu} - \frac{1}{2}\delta^2 t) .$$

The distribution of the second factor is independent of ν — it is a geometric Brownian motion with zero drift — and the first factor obviously converges to infinity as $\nu \to \infty$. Hence for all $M > 0$ we have

[21] The boundedness of the function μ in Assumption 5.1 ensures that for bounded ν_t G defined in (35) actually satisfies $E[G_T] = 1$.

$$\lim_{\nu\to\infty} \lambda^\nu[M,\infty) = 1\,. \tag{51}$$

Combining (49), (50) and (51) now immediately yields $\lim_{\nu\to\infty} E^{Q^\nu}[[X_T - K]^+] = X_0$. Similarly we obtain that for every $M > 0$

$$\lim_{\nu\to-\infty} \lambda^\nu(0,1/M] = 1$$

and hence $\lim_{\nu\to-\infty} E^{Q^\nu}[[X_T - K]^+] = [X_0 - K]^+$. □

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